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Poincare recurrences and topological diversity

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Abstract: Finite entropy thermal systems undergo Poincaré recurrences. In the context of field theory, this implies that at finite temperature, timelike two-point functions will be quasi-periodic. In this note we attempt to reproduce this behavior using the AdS/CFT correspondence by studying the correlator of a massive scalar field in the bulk. We evaluate the correlator by summing over all the $\mathrm{SL}(2,\mathbb{Z})$ images of the BTZ spacetime. We show that all the terms in this sum receive large corrections after at certain critical time, and that the result, even if convergent, is not quasi-periodic. We present several arguments indicating that the periodicity will be very difficult to recover without an exact re-summation, and discuss several toy models which illustrate this. Finally, we consider the consequences for the information paradox.

Keywords: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence, Models of Quantum Gravity, Black Holes.

Non domandarci la formula che mondi possa aprirti,
sì qualche storta sillaba e secca come un ramo.
Codesto solo oggi possiamo dirti,
ciò che non *siamo, ciò che* non *vogliamo.*

http://jhep.sissa.it/archive/papers/jhep102004030/jhep102004030.pdf

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1. Introduction

In this paper, we study the eternal BTZ [1] black hole in 2+1 dimensional AdS space. As discussed in [2]–[5], the field theory dual to this black hole is thermal and generically has a discrete spectrum and finite entropy. In such theories, timelike separated two point correlation functions should be quasi-periodic functions of time. One can think of the correlator $\langle\phi(t)\phi(0)\rangle$ as the insertion of some perturbation at time $t = 0$, and then a measurement of the response at some later time $t > 0$. In thermal field theories, the short-time behavior is an exponential decay due to thermalization, but at long times the quantum version of the Poincaré recurrence theorem guarantees that the correlator will behave stochastically, returning (or at least coming arbitrarily close to) its initial value an infinite number of times.

The bulk dual to the short time exponential fall-off of the correlator is related to the fact that black holes swallow anything that is thrown into them. Classically, perturbed black holes ring with a quasi-normal frequency that has a non-zero imaginary part (so that the oscillation damps to zero exponentially), and semi-classical corrections *á la* Hawking simply create an exactly thermal atmosphere which presumably does not change the qualitative behavior. Therefore, if field theory correlators are computed using free fields in a background BTZ spacetime, the time dependence is a dying exponential, and the recurrences are gone. This is closely related to the information paradox of Hawking [6]. Restoring unitarity and resolving the information paradox, and matching with the CFT prediction, requires that the exact bulk correlator be quasi-periodic.

In this paper we study the behavior of the correlator in an ensemble of 2+1 dimensional bulk spacetimes dual to the boundary CFT. In the AdS/CFT correspondence, we are instructed to sum with an appropriate weight over all bulk geometries that fill a given boundary. This is just the euclidean version of a sometimes forgotten aspect of AdS/CFT: a CFT *state* is dual to a quantum gravity *state*, not to a classical background geometry. Such backgrounds can at best arise as saddle points in a field theory path integral. One problem with this prescription is that, as far as we are aware, it does not arise from any well-defined bulk path integral, and among other problems this means that it is not clear what weight to assign to a given bulk geometry, or even what geometries to include in the sum.

In [7], the authors studied the D1-D5 system, and demonstrated that the field theory elliptic genus, once suitably re-summed, could be interpreted as a sum over the $\mathrm{SL}(2,\mathbb{Z})$ images of the BTZ black hole. One surprising result was that the weight of each geometry was not simply the euclidean action, but rather had an unexpected (from the bulk point of view) additional factor with a definite modular weight.

We begin our analysis in section 2, by reviewing some basic facts and definitions about the BTZ black hole and the problem associated with Poincaré recurrences.

In section 3, we compute the bulk correlator in the same infinite family of geometries that appeared in [7], and study the sum. We show that all but finitely many of the terms receive large perturbative corrections, absent in the index of ref [7]. These corrections are small in the BTZ background and in finitely many of its images under $\mathrm{SL}(2,\mathbb{Z})$, but they become large for all the other terms.

Even if the sum (including these large corrections) converges, the result is *not* quasi-periodic; it suffers from essentially the same difficulty as the result computed simply in the BTZ spacetime. The rest of the paper addresses this question in more detail. In particular, we discuss the role of an interesting critical time at which *all* the contributions from different $\mathrm{SL}(2,\mathbb{Z})$ images of the BTZ metric to the correlator are of the same order, despite the fact that they are exponentially supressed compared to the leading term.

Specifically, in section 4, we exhibit a simple toy model that exhibits part of the difficulty in recovering a quasi-periodic result from an expansion around the classical limit. In this toy model, an exact re-summation of the perturbation series is required to detect the periodicity in time. In our bulk result, we study the interplay of perturbative and non-perturbative corrections arising from the saddle point approximation, and argue that

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such a re-summation is likely to be extremely difficult — perhaps even ill defined. We estimate the back-reaction due to the probe operator itself, and show that this interferes with reliably computing the correlator after the same critical time mentioned above, even in the geometries which do not receive large corrections near $t = 0$.

In section 5, we discuss how to give a bulk spacetime interpretation to Poincaré recurrences.

In section 6, we discuss some of the implications for the information paradox and conclude.

Appendices A, B, C and D detail the derivation of two point functions and the action in the BTZ background as well in its Z_N orbifolds.

2. Poincaré recurrences and Heisenberg time

We begin by briefly reviewing the problem. Consider the Kruskal extension of the AdS-Schwarzschild black hole. This geometry has two AdS boundaries, connected only by spacelike trajectories that pass through the interior of the black hole (see figure 1). This classical geometry suggests that the Hilbert space of the boundary theory is the tensor product of two of the usual Hilbert spaces, with no interaction term in the hamiltonian. Analytic continuation from euclidean space defines a particular bulk state, the thermal field double or Hartle-Hawking state, which is pure but entangles the two sides, and has the property that the expectation value of any product of operators inserted only on one side is equal to the trace of the operators in the thermal ensemble in the original Hilbert space. The theory in this state is completely equivalent to thermal field theory in the ordinary thermal ensemble; in particular, correlators involving operators inserted on both boundaries are simply analytic continuations of ordinary thermal correlators. For a brief review, see [8, 9].

In thermal field theory, finite entropy implies that the spectrum of the hamiltonian is discrete. In such systems, there exist Poincaré recurrences — given an initial configuration,

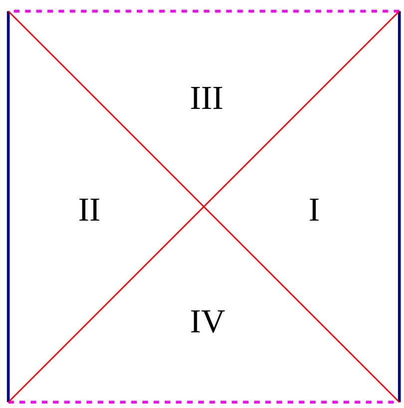

Figure 1: The extended conformal diagram of the BTZ black hole. The regions I and II are outside the black hole, each with its own asymptotic AdS boundary. Regions III and IV are inside the horizon. The two boundaries are connected by spacelike trajectories passing through the horizon.

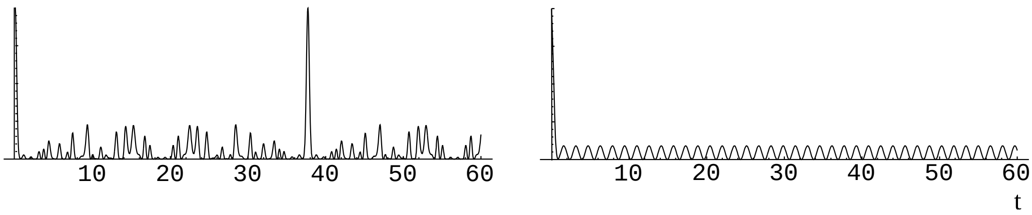

Figure 2: On the right, qualitative behavior of $f(t)$ computed by summing the BTZ black hole and thermal AdS, in the high-temperature regime where the black hole dominates. On the left, a typical quasi-periodic function with the same long time average.

because of the finite phase space volume the system generically evolves under unitary time evolution in such a way that it comes arbitrarily close to its initial state an infinite number of times. While often discussed in the context of classical physics, this phenomenon extends to the behavior of correlators in quantum theories (see [10], the appendix of [11], and [4] for a discussion in this context and additional references). In particular, under some weak assumptions about the operator ϕ, one can prove that the time-like two point function $\langle\phi(t)\phi(0)\rangle \equiv f(t)\langle\phi(0)\phi(0)\rangle$ is an almost periodic function of time,[1] with an average value $(1/T)\int^T |f(t)|^2 dt \sim \exp(-S)$, where S is the entropy in the canonical ensemble. The details of the time dependence of $f(t)$ depend very sensitively on the details of the spectrum, but generically the expected time T between order one recurrences is at least exponentially long in the entropy.

On the other hand, correlators computed in black hole spacetimes are exponentially damped in time, and in particular are *not* almost periodic. They always (in the Hartle-Hawking state, at least) take their maximum value at $t = 0$, and due to the damping never come back to that value again. For the same reason, their time-averaged value is zero. In [2], this second problem was solved by adding another geometry to the calculation of the correlator. In $n+2$ euclidean dimensions, there is at least one euclidean geometry with the same asymptotics as the AdS black hole ($S^1 \times S^n$); namely, global AdS with euclidean time periodically identified. Continuing to lorentzian space gives two copies of AdS in the Hartle-Hawking state. Correlators computed in this geometry will be exactly periodic in time, because of the usual harmonic property of AdS. If we weight the contribution from this geometry by the exponential of its euclidean action, we obtain a correlation function with the correct long-time average (see figure 2).

However, the first problem — that the correlator is not almost periodic — remains [4]. In what follows we compute a more exact correlator by summing over the infinite ensemble of euclidean geometries with the same asymptotics as the BTZ black hole, but we again fail to obtain a quasi-periodic result. The reason is essentially that all of these geometries, with the exception of thermal AdS, contain a black hole, and correlators computed in geometries with horizons always exponentially damp to zero in time. As in [2], the contribution of thermal AdS provides a purely oscillating contribution which remains forever, but with an exponentially suppressed amplitude.

[1] We use the terms almost periodic and quasi-periodic interchangeably here, with the following definition: $f(t)$ is almost periodic if $\forall\, \epsilon > 0$, $\exists\, T$ such that in all intervals I_T of length T, $\exists$ a $\tau \in I_T$ such that $|f(t+\tau) - f(t)| < \epsilon$, $\forall\, t$.

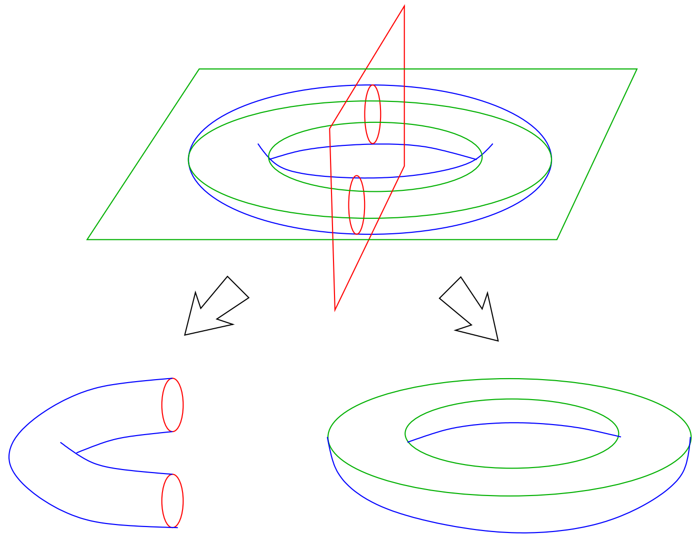

Figure 3: Two possible slicings of a solid torus as $t = 0$ data for Lorenztian space times: on the left the ‘donut slicing’, giving two copies of AdS in an entangled state; on the right the ‘bagel slicing’, giving the BTZ black hole.

3. Topological diversity

For $d = 3$ the conformal boundary topology of the AdS black hole is a two dimensional torus, and there are infinitely many smooth, negative curvature Einstein metrics that can fill a given boundary torus. The saddle point approximation to the correlator is then an infinite sum over these fillings, but as we will show, it is not quasi-periodic.

One way to try to fix this is to include bulk spaces with singularities in the sum. In particular, we will consider 'mild' singularities (orbifolds). These give corrections that can be relevant at large times, but again will not reproduce the recurrences.

3.1 BTZ black holes

We are interested in a 1+1 dimensional thermal field theory on a circle. The thermal condition can be expressed as a periodic compactification of euclidean time, with period $\beta = 1/T$, and with antiperiodic boundary conditions for fermions. This means we should study a euclidean field theory on a torus with periodicities $(\phi, t_E) \to (\phi + 2\pi n, t_E + \beta m)$, $n, m \in Z$, and a flat metric:

$$ds^2 = d\phi^2 + dt_E^2 = (2\pi)^2 |d\sigma_1 + \tau d\sigma_2|^2 \,, \tag{3.1}$$

where $\sigma_1 = \phi/2\pi$, $\sigma_2 = t_E/\beta$ and $\tau = i\beta/2\pi$. In general τ can be complex, with $\mathrm{Im}(\tau) = \beta/2\pi$ the inverse temperature, and $\mathrm{Re}(\tau)$ related to the chemical potential. In what follows, we will be interested in the case $\mathrm{Re}(\tau) = 0$. The coordinates σ_i have periodicity 1.

According to the prescription discussed above, we should look for three dimensional euclidean metrics such that the conformal boundary is a two torus, with a boundary metric in the same conformal class as the one above [12, 7]. One class of such examples is certain rotating BTZ black holes:

$$ds^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2} dt^2 + \frac{r^2 l^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi + \frac{r_+ (ir_-)}{r^2} dt \right)^2 , \qquad (3.2)$$

where l is the AdS radius (in what follows we set $l = 1$). These metrics are parametrized by two real numbers (r_+, ir_-). They describe solid tori, under the identification $z \equiv \phi + it \sim z + 2\pi m + 2\pi\tau' n$, $m, n \in Z$ The conformal boundary is a 2-torus with modular parameter $\tau' = \tau_1' + i\tau_2'$: $r_+ = \tau_2'/|\tau'|^2$ and $r_- = -i\tau_1'/|\tau'|^2$.

To identify the field theory on the torus described by (3.1) with the conformal boundary of the black hole, we need to map the two up to a Weyl transformation; i.e., find a diffeomorphism which maps (σ_1, σ_2) of the field theory to (σ_1', σ_2') of the black hole conformal boundary, such that $ds^2(\tau) = \Omega^2 ds^2(\tau')$. That can be done by an $\mathrm{SL}(2, \mathbb{Z})$ transformation:

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} ,$$
$$\tau' = \frac{a\tau + b}{c\tau + d} . \qquad (3.3)$$

The two point function for a free scalar field of mass m,[2] is (see appendix A):

$$\sum_{n \in Z} \left| 2(a\tau + b) \sin \left[\frac{\phi + it_E + 2\pi n(c\tau + d)}{2(a\tau + b)} \right] \right|^{-4\Delta} , \qquad (3.4)$$

where the conformal weight $2\Delta = 2\bar{\Delta} = 1 + \sqrt{1 + m^2}$. Notice that the two point function is invariant under the transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a & b \\ c + a & d + b \end{pmatrix} . \qquad (3.5)$$

This invariance is a consequence of the fact that, in a solid torus, there is a unique choice of a contractible 1-cycle $[\gamma_1]$. However, the non-contractible cycle $[\gamma_2]$ is determined only up to adding multiples of $[\gamma_1]$; that is, $[\gamma_2] \to [\gamma_2 + n\gamma_1]$ for any integer n. This transformation is generated by eq. (3.5), which is a STS transformation (S: $\tau \to -1/\tau$, T: $\tau \to \tau + 1$). Therefore to avoid overcounting we should sum only over $\mathrm{SL}(2, \mathbb{Z})/Z$, not over the full $\mathrm{SL}(2, \mathbb{Z})$ [7]. In figure 4 we have represented the metrics that contribute to our problem. It is convenient to plot the $-1/\tau$ plane, where the STS symmetry is simply a T-translation. These black holes are parametrized by a pair of co-prime integers (a, b). The integers (c, d) are determined by (a, b) up to an STS transformation.

Presumably, each of these configurations should be weighted by e^{-I}, where I is the euclidean action. The action is proportional to the volume of the space, which is infinite for spaces with an AdS conformal boundary. However, in the context of AdS/CFT we understand how to regulate this divergence. The regularized action is (see appendix D):

$$I = -\pi k r_+ = -\frac{i\pi k}{2} \left(\frac{1}{\tau'} - \frac{1}{\bar{\tau}'} \right) = -\pi k \frac{\tau_2}{|a\tau + b|^2} = -\frac{2\pi^2 k\beta}{a^2\beta^2 + 4\pi^2 b^2} , \qquad (3.6)$$

[2] We measure all masses in units of the AdS radius.

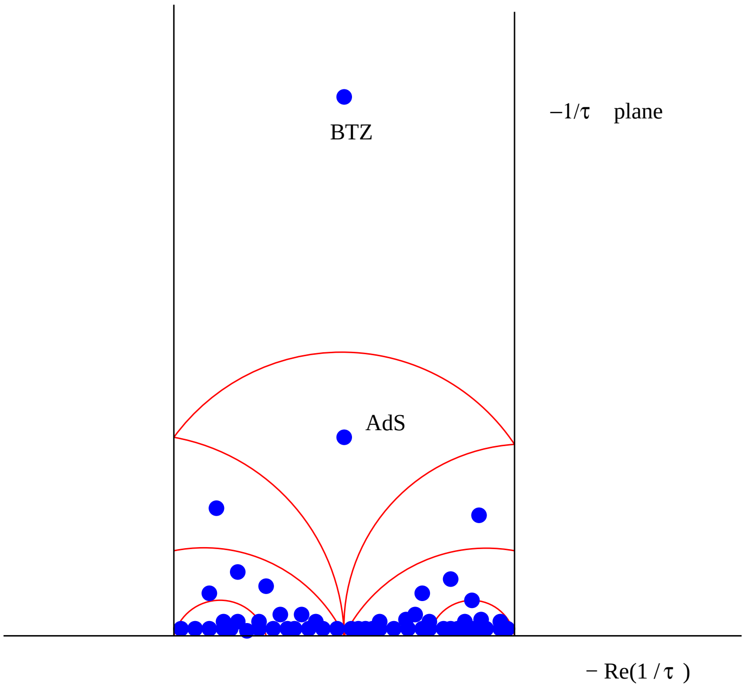

Figure 4: Black holes contributing to our problem are represented as dots in the $-1/\tau$ plane. At high temperature the action of each of these black holes approaches zero as we come close to the real axis.

where $k = c/6 = 1/4G_N$ and c is the central charge of the boundary theory. For small temperatures ($\beta > 2\pi$) the dominant contribution is thermal AdS. For high temperatures ($\beta < 2\pi$) the dominant contribution is the nonrotating BTZ.

We can now construct the correlator in this "ensemble" of geometries, by simply summing the contributions with a weight given by (3.6). However, there is immediately a problem. Consider the part of the sum with $a = 1$, b large. From (3.6), $I \to 0$, so each term has an equal weight. Therefore the partition function will diverge, as will the correlators. Since the action tends to zero in this limit, perturbative corrections can be larger than the classical action. One such correction has been estimated in [13]. It changes the action as

$$I \to I + \delta I\,, \qquad \delta I = -3\log|\tau'|\,. \tag{3.7}$$

It is easy to find $SL(2,\mathbb{Z})$ images of the BTZ black hole where the correction δI is at least as large as the classical action. For instance, take $a = 1, b = n, c = 0, d = 1$. At high temperature, $\beta \ll 1$, the correction becomes of order the classical action when $n^2 \sim \beta k$. For these geometries, $kr_+ \sim 1$, so the black hole radius is at the Planck scale. Such geometries will receive large corrections, which we do not know how to estimate.

If the partition function is to converge, these large b terms must be suppressed. At any rate, we can take this pathology as a signal that whenever subdominant saddles become important for the computation of the two-point function, then the entire semiclassical approximation breaks down.

3.2 Lorentzian correlators

We define the lorentzian two point function by analytically continuing the euclidean one, setting $it_E = t$:[3]

$$e^{\pi k \frac{\tau_2}{|a\tau+b|^2}} \sum_{n \in Z} \left(2|a\tau+b|^2 \sin\left[\frac{\phi + t + 2\pi n(c\tau+d)}{2(a\tau+b)}\right] \sin\left[\frac{\phi - t + 2\pi n(c\bar{\tau}+d)}{2(a\bar{\tau}+b)}\right]\right)^{-2\Delta} . \tag{3.8}$$

This function has poles on the light cone $\phi \pm t = 2\pi N$, where N is an integer.[4] These are the usual divergences one expects when the operators are null separated (there is an infinite number of such points, since the theory is on a circle). They are perfectly physical, but it is more convenient to work with the "two sided" correlator, obtained by continuing $it_E = t + i\beta/2$, which is finite for all times and has no poles:

$$\begin{aligned} e^{\pi k \frac{\tau_2}{|a\tau+b|^2}} \sum_{n \in Z} \Bigg(2|a\tau+b|^2 \sin\left[\frac{\phi + t + i\beta/2 + 2\pi n(c\tau+d)}{2(a\tau+b)}\right] \times \\ \times \sin\left[\frac{\phi - t - i\beta/2 + 2\pi n(c\bar{\tau}+d)}{2(a\bar{\tau}+b)}\right]\Bigg)^{-2\Delta} . \end{aligned} \tag{3.9}$$

For large t, $a \neq 0$, and fixed n the time dependence is:

$$\langle O(t + i\beta/2) O(0)\rangle \sim \exp\left(\pi k \frac{\tau_2}{|a\tau+b|^2}\right) \exp\left(-2\Delta \frac{|a|\tau_2}{|a\tau+b|^2} t\right) . \tag{3.10}$$

(Recall that $\tau_2 = -i\tau = \beta/2\pi$, for the case of zero chemical potential.)

The first exponential comes from the action. The full result is the sum over all geometries with fixed boundary, and hence over all relatively prime integers (a, b). The contribution with $a = 0$ is thermal AdS, and is purely oscillatory. All the other terms are black holes, and have two point functions that decay with time. Notice that this decay can be very slow for configurations with large b. These contributions can be important at very long times because the leading terms decay more quickly, but their amplitude is very small (see figure 5).

As we mentioned above, perturbative corrections to the classical action grow out of control for most of these "spinning" euclidean BTZ black holes, so, whenever these configurations give a significant contribution to the two-point function, the semiclassical approximation breaks down. In this interpretation, the (perturbative) bulk computation of the correlator is not reliable past that critical time.

A very interesting fact is that because of a cancellation between the suppression factor from the action and the slower decay rate, the critical time is the same for all $SL(2, \mathbb{Z})$ images, independently of a and b, namely it is $t_0 \sim \pi k/2\Delta$. To give an interpretation to this time, note that the amplitude of the leading contribution to the correlator has decayed by a factor of $\exp(-S)$ at $t = t_0$. In general, we expect finite entropy effects to become

[3] To avoid unnecessary complications with the continuation, we set 2Δ equal to a positive integer.

[4] We have ignored the $i\epsilon$ prescription here, as we will focus on the "two sided" correlator, which has no lightcone singularities.

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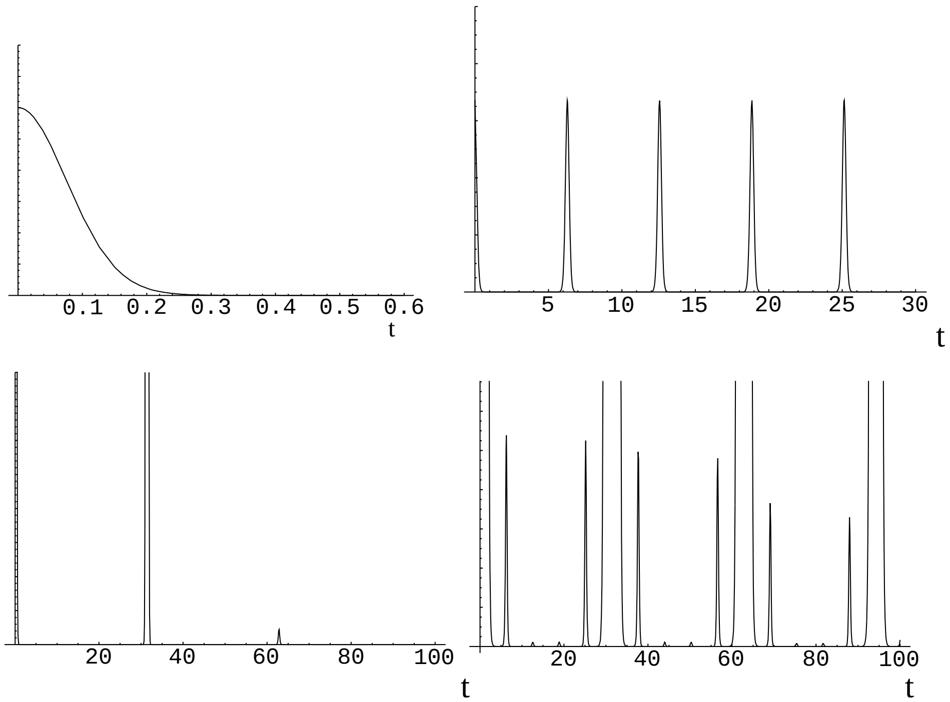

Figure 5: Two point correlators as functions of lorentzian time t for different geometries. We have taken $\beta = 0.95$ and $k = 1$ (larger k would give an exponential weight to the non-rotating BTZ). The upper left figure is the non-rotating BTZ black hole, which is the dominant contribution at high temperature. On the upper right is thermal AdS, which gives an oscillating function with period 2π in AdS units. On the lower left is the $a = 1$, $b = 10$ rotating black hole. There are peaks which become relevant at long times, but the overall contribution is decaying. The lower right figure is a detail of the lower left.

important at times later than t_0, so it is precisely when finite entropy matters that the calculation becomes unrealiable. We will discuss this further later in this section and in section 4. The importance of this time was also noted in [2, 9].

3.3 Orbifold singularities

We have seen that the sum over smooth geometries, even if convergent, gives a decaying two point function at large t, and is unable to reproduce the expected quasi-periodicity. One possible fix is to add some singular bulk geometries. Without understanding the structure of the theory better, we do not know which geometries should contribute. Our only guide is the calculation of [7], so we will follow that work and consider euclidean geometries that have an orbifold singularity at some point in the bulk (but of course with the same boundary torus).[5]

[5]Ref. [7] considered a particular index that receives no contribution from black hole states, so one may question the wisdom of focusing on the same class of bulk metrics that were considered there. Here, we only use [7] as a justification for including certain singular metrics that would be discarded in a naive semiclassical approximation to quantum gravity.

If we allow a conical deficit angle $2\pi(1-1/N)$, with N a natural number, the solution can be constructed easily by taking a Z_N identification. In AdS, this is simply the usual way to define a point particle, where the mass is proportional to the deficit angle:

$$M = \frac{1}{8G_3}\left(1-\frac{1}{N}\right). \tag{3.11}$$

In appendix C we compute the two point function of these euclidean orbifolds, and in appendix D we compute their action. The euclidean two point function, including the action, is:

$$\sum_{n=1}^{N}\sum_{m\in Z}\kappa \exp\left(\frac{\pi k\tau_2}{|a\tau+b|^2N^2}\right)\left|2N(a\tau+b)\sin\left[\frac{u+2\pi m(c\tau+d)+2\pi n(a\tau+b)}{2N(a\tau+b)}\right]\right|^{-4\Delta}. \tag{3.12}$$

For $it_E = t+i\beta/2$:

$$\begin{aligned}\left\langle O\left(t+\frac{i\beta}{2}\right)O(0)\right\rangle &= \sum_{n=1}^{N}\sum_{m\in Z}\kappa\ \exp\left(\frac{\pi k\tau_2}{|a\tau+b|^2N^2}\right)(2N)^{-4\Delta}|a\tau+b|^{-4\Delta}\times \\ &\quad\times\left(\sin\left[\frac{\phi+t+i\beta/2+2\pi m(c\tau+d)+2\pi n(a\tau+b)}{2N(a\tau+b)}\right]\times\right. \\ &\quad\left.\times\sin\left[\frac{\phi-t-i\beta/2+2\pi m(c\bar\tau+d)+2\pi n(a\bar\tau+b)}{2N(a\bar\tau+b)}\right]\right)^{-2\Delta}.\end{aligned} \tag{3.13}$$

The long time behavior ($a\neq 0$) of these functions is:

$$\langle O(t+i\beta/2)O(0)\rangle \sim \exp\left(-\frac{|a|\tau_2}{|a\tau+b|^2}t\right). \tag{3.14}$$

Naively, the correlator decays as $2\Delta t/N$ rather than t as above, but one can easily check that, thanks to the sum over the Z_N images, all contributions decaying as $\exp[-(2\Delta+l)(|a|\tau_2/|a\tau+b|^2)t/N)]$ vanish for all integers $0<l<N-2\Delta$. Therefore, Z_N orbifolds contribute terms to the two-point function that decay in time much more slowly (by a factor of 2Δ) than in a rotating black hole background [eq. (3.10)]. However, the action for them differs by a factor of $1/N^2$ (see appendix D), so their amplitude is suppressed. As in the case of the $\mathrm{SL}(2,Z)$ images, this implies that there is a critical time t_0 where the contributions from these are of the same order as the leading term. Including the oribifold geometries once again fails to reproduce the correct time dependence.

Specializing to the case of the non-rotating black hole ($\tau = i\beta/2\pi$), the terms in the two-point function coming from the Z_N orbifold geometries will be of order the contribution of the black hole at a time t_0 satisfying $2\pi t_0/\beta + I_{BH}/N^2 = 4\pi\Delta t_0/\beta + I_{BH}$, which yields (using $I_{BH} = -2\pi^2 k/\beta$ and taking N, Δ large):

$$t_0 \sim \frac{\pi k}{2\Delta}, \tag{3.15}$$

which is also the time where the $\mathrm{SL}(2,Z)$ images of the black hole make order one contributions. So at $t=t_0$, *every geometry considered here makes a contribution of the same*

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order to the two point function. A crucial point is that since all the terms take their global maximum at $t = 0$, it is impossible for the result to be quasi-periodic, even if the corrections are such that the sum converges.

4. Large entropy limit

The entropy of a conformal field theory is proportional to the central charge. The limit $c = 6k \to \infty$ corresponds to $G_N \sim 1/k \to 0$. The infinite entropy limit of the field theory is the classical gravity limit of the bulk (and of course is also the limit where the gravitational entropy diverges). We will argue that in order to recover Poincaré recurrences, we need information about the entire expansion in k^{-1}. In general the series will contain both perturbative, i.e. $O(1/k^n)$, and non-perturbative, $e^{-\alpha k}$ terms. In this section we will consider some simple examples that illustrate how large entropy limits behave.

4.1 A toy model

Consider a particle on a circle of radius $R = k$. The spectrum is discrete, with $\Delta E = 1/k$. In the limit $k \to \infty$, the spectrum becomes continuous and the entropy infinite. A two point function in the thermal ensemble will be of the form:

$$f(t) \equiv \langle O(t)O(0)\rangle = \sum_n \frac{e^{-\beta E_n}}{Z(\beta)} \langle n|O(t)O(0)|n\rangle = \sum_{n,m} \frac{e^{-\beta E_n}}{Z(\beta)} e^{it(E_n - E_m)} |O_{nm}|^2 \,, \tag{4.1}$$

where $O_{nm} \equiv \langle n|O(0)|m\rangle$. The energies are $E_n = |n|/k$ for every integer n, and hence:

$$Z(\beta) = \frac{1 + e^{-\beta/k}}{1 - e^{-\beta/k}} \,. \tag{4.2}$$

To take a simple example, consider an operator O such that $|O_{nm}|^2 = \delta_{n,0} + \delta_{m,0}$, then

$$Z(\beta) f(t - i\beta/2) = 1 + \frac{\sinh(\beta/2k)}{\cosh(\beta/2k) - \cos(t/k)} \,, \tag{4.3}$$

where we have regulated the divergences by taking $t \to t - i\beta/2$. This is a manifestly periodic function of t, with period k.[6] However, consider the expansion around the continuous spectrum at $k = \infty$:

$$\begin{aligned} f(t - i\beta/2) \; = \; & \frac{2\beta^2}{4t^2 + \beta^2} + \frac{\beta}{2k} + \frac{\beta^2(4t^2 - 3\beta^2)}{k^2(96t^2 + 24\beta^2)} + \frac{\beta^3}{24k^3} + \\ & + \frac{\beta^2(16t^4 - 24t^2\beta^2 + 25\beta^4)}{1920k^4(4t^2 + \beta^2)} + \mathcal{O}\left(\frac{1}{k^5}\right) . \end{aligned} \tag{4.4}$$

In the limit of infinite entropy $f(t) \sim 2\beta^2/(4t^2 + \beta^2)$, and the function is not periodic. To recover the exact periodicity we would need to resum the entire series. It is evident that as t increases, one must keep more and more terms in the sum to get an accurate estimate of

[6] The function is exactly periodic because of the simple form of the spectrum; interactions will shift the energy levels and generically make the time dependence quasi-periodic.

the value of the function. For this simple model, the power series is absolutely convergent for any t, so this is possible. One could determine with greater and greater confidence that the function is periodic, by keeping more and more terms in the $1/k$ expansion. However, the behavior is quite different in an expansion around a saddle, where if there are other saddles contributing, the series is asymptotic (and the divergent part of the series, when appropriately re-summed, gives information about the sub-leading saddles [14]). In particular, there is a limit to the accuracy that can ever be obtained by expanding only around one saddle.

4.2 Winding string

A more interesting example, that has been analyzed in [2], is a field $X(\sigma,\tau)$ that lives on a circle of radius $R \sim k$. This field theory is a simple model of the long string model of the black hole of [15]. In [2] the two point correlator of the operator $O = \sum_n \partial X \bar{\partial} X(\tau, \sigma+2\pi n)$ was computed:

$$\langle O(\phi,t)O(0,0)\rangle \sim \sum_{n,m} \left[\cosh\frac{2\pi(t+2\pi mk)}{\beta} + \cosh\frac{2\pi(\phi+2\pi n)}{\beta}\right]^{-2}. \tag{4.5}$$

The function is periodic in time with period k. For $\phi = 0$ and times $|t| < \pi k$ the leading term in the large k limit is:

$$\langle O(0,t)O(0,0)\rangle \sim \sum_{n} \left[\cosh\left(\frac{2\pi t}{\beta}\right) + \cosh\left(\frac{4\pi^2 n}{\beta}\right)\right]^{-2}. \tag{4.6}$$

The correlation function decays exponentially with time, with a decay rate given by the temperature. The corrections are of the form $e^{-4\pi^2|m|k/\beta}$:

$$\begin{aligned}\delta\langle O(0,t)O(0,0)\rangle \sim \sum_{n,m>0} e^{-8\pi^2 mk/\beta}\Bigg[\left(e^{2\pi t/\beta} + e^{-4\pi^2 mk/\beta}\cosh\left(\frac{4\pi^2 n}{\beta}\right)\right)^{-2} + \\ + \left(e^{-2\pi t/\beta} + e^{-4\pi^2 mk/\beta}\cosh\left(\frac{4\pi^2 n}{\beta}\right)\right)^{-2}\Bigg].\end{aligned} \tag{4.7}$$

4.3 Breakdown time scale

We have seen that at the critical time $t_0 \sim \pi k/2\Delta$, all the geometries we are considering make a contribution of the same order to the two point function. Close to $t = 0$, on the other hand, the amplitude of the black-hole contribution is exponentially larger. One way to think about the Z_N orbifold geometries, as discussed in [7], is that they represent the back-reaction of Hawking particles on the geometry. The conical defects are interpreted as virtual particles near the horizon. Keeping for the moment only two geometries, the BTZ black hole and its Z_N orbifold for $N \gg 1$, we obtain the ratio (for large t):

$$\frac{\langle\phi(t)\phi(0)\rangle_{BH} + \langle\phi(t)\phi(0)\rangle_{Z_N}}{\langle\phi(t)\phi(0)\rangle_{BH}} \sim 1 + e^{(4\pi\Delta t - 2\pi^2 k)/\beta}\,. \tag{4.8}$$

If we can regard the Z_N geometries as representing the effect of perturbations around the black hole, we can see clearly from this equation that the perturbations become large at the critical time t_0.[7] This gives an indication that we can not trust the semi-classical approximation for $t > t_0$, even in the finite number of geometries where the black hole is larger than string scale. Since in fact *all* the relevant geometries are of the same order at this time, an honest calculation would require taking them all into account, including perturbative corrections around them. We will discuss this further in the next section.

5. Beyond gravity

There are various ways we could imagine addressing the mystery of the missing recurrences. The most radical is to rethink the meaning of the AdS/CFT "duality." It could be that only the CFT side of the correspondence is well-defined as a quantum theory, and that *any* bulk description, including the one given by perturbative, closed string theory is incomplete. A more modest goal is instead to find a "phenomenological" description of Poincaré recurrences in AdS gravity language. Precisely, the question would not be how to predict the Poincaré recurrences in AdS, but rather to ask what the bulk spacetime looks like on time scales of order the recurrence time.

5.1 "Phenomenological" description of recurrences

One option is to modify the state in which we are computing the correlator. Continuation from euclidean space defines a unique lorentzian correlator, but if we modify the state in the lorentzian spacetime away from the Hartle-Hawking state, we can cook up more complicated time dependences. Normally such modifications lead to divergent terms in the expectation value of the stress tensor at the horizon, and as such are inconsistent. However, it is possible to avoid this, as we demonstrate below.

Consider for simplicity a non-rotating BTZ black hole. It has a single horizon: $r_+ = r_H, r_- = 0$. A generic scalar Green function that depends only on $t-t'$ and $\phi-\phi'$ reads:

$$G(t,\phi,r|t',\phi',r') = G_{HH}(t-t',\phi-\phi',r,r') + \\ + \sum_m \int d\omega H_m(\omega)\bar{F}_m(\omega,r)F_m(\omega,r')e^{-i\omega(t-t')+im(\phi-\phi')}. \quad (5.1)$$

Here, $F_m(\omega,r)\exp(im\phi - i\omega t)$ denotes a normalizable solution of the wave equation, with frequency ω and angular momentum m. The explicit form of the solution can be found for instance in [16]. Near the boundary it behaves as $F_m(\omega,r) \approx r^{-2\Delta}$. Near the horizon it becomes [16, 17]

$$e^{-i\omega t+im\phi}F_m(\omega,r) \approx e^{-i\omega t+im\phi}\left[e^{i\omega(\beta/4\pi)\log(r-r_H)} + e^{2i\theta(\omega)-i\omega(\beta/4\pi)\log(r-r_H)}\right]. \quad (5.2)$$

The function $\exp[2i\theta(\omega)]$ has poles at

$$\omega = inr_H\,, \quad n=1,2,3,\dots\,; \qquad \omega = \pm m - 2inr_H - 2ir_H\Delta\,, \quad n=0,1,2,\dots\,. \quad (5.3)$$

[7]The reader may wonder where there is room in the usual computation of Hawking radiation for such order one effects. The answer is presumably that the quantity that is being corrected is already exponentially small.

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Here, we used eqs. (3.17,3.18) of ref. [18], and we set $r_- = 0$, $r_+ = r_H$, $\Lambda = 1$, $h_+ = \Delta$. The Green function G_{HH} can be used to compute the expectation value of the stress-energy tensor on a black-hole geometry, $\langle T_{\mu\nu}\rangle$ which is finite on both future and past horizons. We want to preserve this property, so we have to make sure that the additional contribution to $\langle T_{\mu\nu}\rangle$ in eq. (5.1) is also well behaved. For the trace of the stress-energy tensor, the contribution can be written as

$$\langle \delta T^\mu_\mu \rangle \approx \sum_m \int d\omega H_m(\omega)\omega^2 \frac{\beta}{4\pi}\left[e^{2i\theta(\omega)+(-1-i\omega/r_H)\log(r-r_H)} + \text{c.c.}\right]. \qquad (5.4)$$

For $r - r_H \ll 1$, the integral of the first term in brackets is evaluated by closing the contour in the upper half-plane. If the poles of $H_m(\omega)$ satisfy $\operatorname{Im}\omega > r_H$, then, thanks to eq. (5.3), the integral is finite. The *c.c.* term in the integral is evaluated by closing the contour in the lower half-plane, resulting in the condition $|\operatorname{Im}\omega| > r_H$ [cfr. eq. (3.19) in [18]].

Notice that when $t - t' \gg 1$, the Green function eq. (5.1) always decays exponentially. On the other hand, if $H_m(\omega)$ has a pole of order $N + 1$ at ω_0, then the Green function behaves as $(t - t')^N \exp(-\operatorname{Im}\omega_0|t - t'|)$, so it grows then decays.

Of course, this *ad hoc* modification of the lorentzian Green function cannot predict the recurrence time, but at best describe it. At this point it may be useful to draw a parallel with thermodynamics. Thermodynamics can describe the evolution of macrostates obtained by averaging over many macroscopically indistinguishable microstates. It can be used reliably to describe the approach of a macroscopic system to equilibrium, but not to predict Poincaré recurrences. Likewise, the bulk description of AdS gravity may be able to predict macroscopic properties of spacetime (such as Hawking radiation), but not its fine details: recurrences, the encoding of information in Hawking radiation, the discreteness of the dual CFT spectrum and so on. In this view, the best that we can do is try to give a "phenomenological" bulk description of features fully describable only in the dual CFT language. The *ad hoc* modification to the two-point function is just such an attempt. It shows that a Poincaré recurrence can be described as a small deviation from the thermal vacuum at $t = 0$ that evolves in time in such a way as to produce a resurgence at some later time. In this description, spacetime is always close to BTZ, since the initial state was chosen to have negligible backreaction on the metric.

A different yet related possibility is to allow the backreaction to become big at some intermediate time. This signals a macroscopic departure of the metric from the BTZ background. This possibility should not be discarded, since the recurrence time is much bigger than $t \sim \pi k/2\Delta$ — that is, the average time of a large fluctuation that converts a BTZ black hole into a thermal gas of light particles.

Finally, one could simply cut off the horizon with a 't hooftian "brick wall." As explained nicely in [4], such a wall acts as an IR cutoff in the effective Schrödinger problem for bulk fields, quantizing their spectrum, making the entropy finite, and hence necessarily leading to a (quasi) periodic result for the bulk two point correlator. However, this is highly unnatural from the point of view of the euclidean geometries. Euclidean thermal AdS and the BTZ black hole have identical topologies, and cutting off the black hole horizon (which is simply the origin in the euclidean space) would correspond to arbitrarily cutting out a tube in the thermal AdS.

5.2 Asymptotics

There is another option, which unfortunately is very difficult to study without a better understanding of the path integral. The asymptotic expansions of one-dimensional integrals can exhibit a variety of interesting behaviors, known collectively as Stokes' phenomena. In particular, as a function of some parameter in the integrand, the steepest-descent integration contour can change discontinuously, causing sub-dominant saddle points to appear or disappear (this typically occurs along a co-dimension one line in complex parameter space, known as a Stokes' line). Further deformation of the parameter can cause a subdominant saddle to become dominant (an anti-Stokes' line; see [14] for a concise and elegant discussion).

There is an instructive example of such a phenomenon in a situation related to the one studied here. In the AdS black hole in $d \geq 4$, there is a family of spacelike geodesics connecting points on the two disconnected boundaries, which have the bizarre property that they asymptote to a null geodesic for special pairs of points. As discussed in [19], this naively leads to the (impossible) conclusion that the spacelike separated two-point function has a lightcone divergence. However, after a careful analysis of the euclidean continuation, this apparently dominant saddle turns out to be off the contour of integration in the relevant region of parameter space (the parameter here is simply lorentzian time), and therefore does not contribute at all. The naive saddle point approximation gives a totally wrong result.

This leads to another possibility for recovering the recurrences. Suppose that not all the various bulk geometries contribute to the calculation of the two-point function for all values of t. For example, for small t only the BTZ might contribute, and then for larger and larger values of t more contributions might appear. One could cook up a scenario in which the two point function is quasiperiodic, by judiciously adding in more and more geometries at late times. This would probably require a weight for the individual geometries in the sum so that the partition function is divergent, if summed over all the contributions. More seriously, it is again completely *ad hoc* without a much more detailed understanding of the gravitational path integral.

6. Conclusions

In this paper we attempted to reproduce the long time dependence of the two point correlator in a finite entropy thermal field theory by computing bulk correlators in an infinite number of geometries with fixed conformal boundary. We have shown that the summation over the geometries that appeared in the calculation of the elliptic genus in [7] suffers from perturbative corrections that grow large for the "spinning" euclidean black holes. These corrections nullify any attempt to use semiclassical bulk computations to compute Poincaré recurrences, which occur at times exponentially longer that the time $t_0 \sim \pi k/2\Delta$, at which the "bad" saddle points become significant. We have identified two forms of corrections to the correlator in these geometries — first, that all but finitely many of the $SL(2,\mathbb{Z})$ images contain black holes smaller than the string/Planck scale, and have leading corrections that render the $SL(2,\mathbb{Z})$ sum divergent, and second, that even in the geometries with large black holes, the back-reaction effects of the probe become of order unity at $t = t_0$. It is interesting that the time t_0 is also the time when an observer would first begin to detect finite entropy effects in the form of exponentially small fluctuations. Reproducing the recurrences would probably require an exact re-summation of the perturbative and non-perturbative contributions to the sum, and perhaps an understanding of the integration contours and generalized Stokes' phenomena as well.

To some of us, the most intriguing question which arises in the consideration of these issues is the following: if gravity emerges only as a saddle point in some path integral, do we expect an observer living in a large, weakly curved space containing a black hole to measure finite entropy effects in Hawking radiation? "Non-thermalities" (that is, finite entropy effects) in the spectrum of Hawking particles will always be invisible if one expands only around the leading saddle point. The sum over geometries may not be enough to restore unitarity and resolve the information paradox, but it is both necessary and predicted by the AdS/CFT correspondence.

Large black holes in AdS are thermodynamically stable, that is, they have positive specific heat. However, if their entropy is finite as is expected from the Gibbons-Hawking calculation, they should undergo Poincaré recurrences. Such events would be forbidden in classical gravity, because they will involve a decrease in the horizon area, but will be consistent with global conservation laws. One possible (but exponentially unlikely) event is that the black hole completely dissolves into particles, which in an AdS time form a black hole again. Such an intermediate state would look similar to thermal AdS in the unstable (high temperature) phase. It is then very tempting to consider the thermal AdS contribution to the correlator as representing this process. Another process that would signal departure from perfect thermal equilibrium was described in section 5: a small deviation from equilibrium at time $t = 0$ focuses at later times to produce a large fluctuation in the two-point function of a certain observable. Both effects could be measured quite easily by an observer in the AdS Schwarzschild space, but it will require a much better understanding of the theory before we can address these issues definitively.

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A. Two Point Function in the BTZ Metric

In the usual metric for the hyperbolic plane

$$ds^2 = \frac{dz^2 + dw d\bar{w}}{z^2}\,, \tag{A.1}$$

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one gets the correlation function for a field of conformal weight $(\Delta, \bar{\Delta})$:

$$\langle O(w)O(w')\rangle = \kappa\,(w-w')^{-2\Delta}(\bar{w}-\bar{w}')^{-2\bar{\Delta}}\,, \tag{A.2}$$

where κ is a constant that it is irrelevant to our discussion.

Here we will consider operators associated to scalar particles in the bulk, i.e. $2\Delta = 2\bar{\Delta} = 1+\sqrt{1+(ml)^2}$, where m is the mass of the particle and l is the AdS radius (from now on, we will set $l=1$). For particles of mass $m \gg 1$, $2\Delta \sim m+1+O(1/m)$, i.e. the mass of the bulk particle and the conformal weight of the dual operator are approximately equal.

To avoid unnecessary complications with the analytic continuation to lorentzian signature we set 2Δ equal to a natural number.

The two-point function in the black hole geometry is written in terms of the torus coordinate $u = (\phi + it_E)$, related to w by $w = \exp[iu/(a\tau+b)]$.

Under a change of coordinate $w = f(u)$, a conformal correlator changes as

$$\langle O_{\mathrm{new}}(u)O_{\mathrm{new}}(u')\rangle = \left|\frac{f'(u)f'(u')}{(f(u)-f(u'))^2}\right|^{2\Delta}. \tag{A.3}$$

In our case $f'(u) = i(a\tau+b)^{-1}f(u)$; so the correlator is:

$$\left|2(a\tau+b)\sin\left[\frac{u+2\pi n(c\tau+d)}{2(a\tau+b)}\right]\right|^{-4\Delta}. \tag{A.4}$$

In terms of the coordinate w, the identification that gives rise to the black hole metric is[8]q

$$w \sim w\exp\left(\frac{2\pi i}{\tau'}\right), \tag{A.5}$$

where $\tau' = (a\tau+b)/(c\tau+d)$ and for our problem $\tau = i\beta/2\pi$. Then the sum over images gives:

$$\sum_{a,b,c,d\in\Gamma_\infty\Gamma, n\in Z} \kappa\,\exp(-I-\delta I)\left|2(a\tau+b)\sin\left[\frac{u+2\pi n(c\tau+d)}{2(a\tau+b)}\right]\right|^{-4\Delta}, \tag{A.6}$$

where I is the classical action:

$$I = -\frac{i\pi k}{2}\left(\frac{1}{\tau'}-\frac{1}{\bar{\tau}'}\right) = -\pi k\frac{\tau_2}{|a\tau+b|^2}\,, \tag{A.7}$$

and δI represents peturbative corrections, for instance as in eq. (3.7).

[8] In the usual BTZ (r, ϕ, t_E) coordinates:

$$w = \left(\frac{r^2-r_+^2}{r^2-r_-^2}\right)^{1/2} e^{i(\phi+it_E)/(a\tau+b)}\,,$$
$$z = \left(\frac{r_+^2-r_-^2}{r^2-r_-^2}\right)^{1/2} e^{r_+\phi+ir_-t_E}\,,$$

where $r_+ = \tau_2/|\tau|^2$ and $r_- = -i\tau_1/|\tau|^2$.

B. SL$(2, \mathbb{C})$

The infinitesimal isometries of the metric eq. (A.1) are:

$$w \to \alpha + \beta w + \gamma w^2 - \bar{\gamma} z^2 \,, \qquad z \to (\mathrm{Re}\,\beta + \gamma w + \bar{\gamma}\bar{w}) z \,, \qquad \alpha, \beta, \gamma \in C \,. \tag{B.1}$$

They realize an SL$(2, \mathbb{C})$ that takes the boundary $z = 0$ into itself, and there it acts on w in the usual manner:

$$w \to \frac{Aw + B}{Cw + D} \,, \qquad A, B, C, D \in C \,, \qquad AD - BC = 1 \,. \tag{B.2}$$

Up to conjugations in SL$(2, \mathbb{C})$, any element of SL$(2, \mathbb{C})$ can be reduced to one of the following two forms:

$$\begin{pmatrix} \exp\theta & 0 \\ 0 & \exp(-\theta) \end{pmatrix} , \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{B.3}$$

In particular, the first form gives both thermal AdS ($\theta = \beta$) and the BTZ black hole ($\theta = 4\pi^2/\beta$).

Now choose two commuting elements in SL$(2, \mathbb{C})$, T_1, T_2, and consider the discrete group $G = \{nT_1 + mT_2, m, n \in Z\}$. We want to sum over all geometries that are asymptotically a torus, so the general question is, when is C/G a torus? Up to now, we considered only one case, that gives both thermal AdS and the euclidean BTZ black hole:

$$T_1 = \begin{pmatrix} \exp(2\pi i/\tau) & 0 \\ 0 & \exp(-2\pi i/\tau) \end{pmatrix} , \qquad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{B.4}$$

We can also consider the torus generated by two non-collinear translations in the $(w, \bar{w})$ plane. They give the extremal black hole at nonzero temperature. This configuration is singular, but our attitude is that we should allow for such configurations; judiciously. However, it is not difficult to see that none of the tori generated by two translations give any significant contribution to the long-time behavior of the two-point function.

The most general group G is obtained as follows. Using a conjugation in SL$(2, \mathbb{C})$, T_1 can be cast in the form given in eq. (B.3). Now, write T_2 as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{B.5}$$

and impose that it commutes with T_1. This gives us the following cases:

1. When T_1 is the identity, T_2 is arbitrary, but we can still use conjugations in SL$(2, \mathbb{C})$ (they commute with T_1 of course!) to recast it in either diagonal or upper-triangular form. In the first case, if $\theta \neq 0$ we get the identifications of BTZ or AdS type. If $\theta = 0$, C/G is not a torus. When T_2 is upper-triangular, we do not get a torus from the quotient.

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2. When T_1 is upper triangular, T_2 is also upper triangular of the form:

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}. \tag{B.6}$$

Iff $\mathrm{Im}\, B \neq 0$ we get a torus.

3. When T_1 is diagonal with $\theta \neq 0$, then T_2 is also diagonal

$$\begin{pmatrix} \exp(\vartheta) & 0 \\ 0 & \exp(-\vartheta) \end{pmatrix}. \tag{B.7}$$

When $\theta = \vartheta$ we get a torus, because the angular coordinate in z is automatically compact. When $\theta \neq \vartheta$ we must have either $\theta = 2\pi ik/N$ or $\vartheta = 2\pi ik/N$ to get a torus. This is the point-particle case and its $\mathrm{SL}(2, Z)$ images.

C. Z_N orbifolds and two point functions

Here we will compute the two point functions for Z_N orbifolds of the BTZ black hole. As BTZ is itself a Z orbifold of global AdS, we can construct these spaces with a hyperbolic quotient (the BTZ quotient or thermal AdS) and an elliptic quotient of finite order.

Let us take the point particle in thermal AdS, and then construct all the others by $\mathrm{SL}(2, \mathbb{Z})$ transformation. We consider the two identifications:

$$w \sim \exp\left(\frac{2\pi i}{N}\right) w\,, \qquad w \sim \exp\left(\frac{\beta}{N}\right) w = \exp\left(-\frac{2\pi i\tau}{N}\right) w\,. \tag{C.1}$$

The first identification yields the Z_N orbifold in the angular coordinate and the second one the finite temperature identification of thermal AdS. To get an angular coordinate with the usual periodicity we have to take $w = e^{iu/N}$, $u = \phi + it_E$.

Using the standard rules for transforming conformal fields as in appendix A, and summing over images, we get

$$\langle O_{\rm new}(u) O_{\rm new}(u')\rangle = \kappa \sum_{n=1}^{N} \sum_{m \in Z} \left| 2N \sin\left(\frac{u + 2\pi(n - m\tau)}{2N}\right)\right|^{-4\Delta}. \tag{C.2}$$

The lorentzian continuation is $it_E \to t + \pi\tau$; then the two-point function at $\phi = 0$ becomes

$$\begin{aligned} \langle O_{\rm new}(t) O_{\rm new}(0)\rangle \, = \kappa \sum_{n=1}^{N} \sum_{m \in Z+1/2} \Bigg| (2N)^2 \sin\left(\frac{t + 2\pi(n - m\tau)}{2N}\right) \times \\ \times \sin\left(\frac{-t + 2\pi(n - m\bar\tau)}{2N}\right)\Bigg|^{-2\Delta}. \end{aligned} \tag{C.3}$$

Notice that for $n = 0$, $m = 1/2$, $t = 2\pi N$, and $N \gg 1$ the two-point function is $O(1)$

$$\langle O_{\rm new}(2\pi N) O_{\rm new}(0)\rangle \approx \left(\frac{2}{\beta}\right)^{2\Delta}. \tag{C.4}$$

The generalization to the BTZ black holes is straightforwardly obtained by applying $\mathrm{SL}(2,\mathbb{Z})$ transformations to eq. (C.4), in complete analogy with the BTZ case worked out in appendix A. In euclidean time the two point function is thus:

$$\sum_{n=1}^{N}\sum_{m\in Z}\kappa\exp(-I)\left|2N(a\tau+b)\sin\left[\frac{u+2\pi m(c\tau+d)+2\pi n(a\tau+b)}{2N(a\tau+b)}\right]\right|^{-4\Delta}, \tag{C.5}$$

It can be analytically continued to lorentzian signature by the same substitution we used earlier: $it_E\to t+\pi\tau$.

D. Regularized action for the BTZ black holes

In this appendix we evaluate the regularized action for the euclidean BTZ black hole by using the method of counterterms [20, 21, 22]. This method is based on the fact that all the divergences that appear in the Einstein-Hilbert action, plus the Gibbons-Hawking term, can be cancelled by a finite set of boundary integrals. In three dimensions we have to sum three contributions:

$$I_{\mathcal{M}} = -\frac{1}{16\pi G_N}\int_{\mathcal{M}}\sqrt{g}\left(R+\frac{2}{l^2}\right), \tag{D.1}$$

$$I_{\partial\mathcal{M}} = -\frac{1}{8\pi G_N}\int_{\partial\mathcal{M}}\sqrt{h}K\,, \tag{D.2}$$

$$I_c = \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}}\sqrt{h}\,, \tag{D.3}$$

where g_{ij} is the 3 dimensional BTZ metric, l is the AdS radius, K is the trace of the extrinsic curvature, and h_{ij} is the induced metric. The first contribution is the usual Einstein-Hilbert action for AdS spaces, regulated by putting a cut-off surface $\partial\mathcal{M}$ at the radius $r=l/\epsilon$. The second contribution is the Gibbons-Hawking boundary term at the cut-off, and the last term is the counterterm.

By explicit evaluation with the metric eq. (3.2) we obtain:

$$I_{\mathcal{M}} = \frac{\beta}{4G_N}\frac{1}{\epsilon^2}\left(1-\epsilon^2\frac{r_+^2}{l^2}\right), \tag{D.4}$$

$$I_{\partial\mathcal{M}} = -\frac{\beta}{4G_N}\frac{1}{\epsilon^2}\left[1+\left(1-\epsilon^2\frac{r_+^2}{l^2}\right)\left(1-\epsilon^2\frac{r_-^2}{l^2}\right)\right], \tag{D.5}$$

$$I_c = \frac{\beta}{4G_N}\frac{1}{\epsilon^2}\sqrt{\left(1-\epsilon^2\frac{r_+^2}{l^2}\right)\left(1-\epsilon^2\frac{r_-^2}{l^2}\right)}. \tag{D.6}$$

Summing the three contributions and taking the $\epsilon\to 0$ limit we find:

$$I_0 = \frac{\beta}{8G_N}\frac{r_-^2-r_+^2}{l^2}\,. \tag{D.7}$$

Using the relation between the temperature, r_+, and r_-, we have:

$$I_0 = -\frac{\pi r_+}{4G_N} = -\pi k\frac{r_+}{l}\,. \tag{D.8}$$

If we are interested in a euclidean non-rotating BTZ black hole with a conical deficit at the horizon then we have to include a contribution from the curvature singularity (see [23]):

$$I_{\text{sing}} = -\frac{1}{16\pi G_N}\int_{\text{sing}} \sqrt{g}R = -\frac{r_+}{4G_N}\delta\,, \tag{D.9}$$

where δ is the deficit angle $\delta = 2\pi - \beta r_+/l^2$. The total gravitational action is then:

$$I_g \equiv I_{\text{sing}} + I_0 = \frac{1}{8G_N}\frac{\beta r_+^2}{l^2} - \frac{\pi r_+}{2G_N}\,. \tag{D.10}$$

It is interesting to write this action in terms of β and the deficit angle:

$$I_g = -\frac{\pi^2}{2\beta G_N}\left[1-\left(\frac{\delta}{2\pi}\right)^2\right]. \tag{D.11}$$

If we interpret this conical deficit as created by a euclidean "point particle" of mass m we have to include its action:

$$I_{pp} = mL = \frac{\pi^2}{\beta G_N}\frac{\delta}{2\pi}\left[1-\left(\frac{\delta}{2\pi}\right)\right]. \tag{D.12}$$

And then the total action is:

$$I_{\text{tot}} = I_g + I_{pp} = -\frac{\pi^2}{2\beta G_N}\left[1-\left(\frac{\delta}{2\pi}\right)\right]^2 = -\frac{\beta}{8G_N}\frac{r_+^2}{l^2}\,. \tag{D.13}$$

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